A NOTE ON THE TOTAL NUMBER OF CYCLES OF EVEN AND ODD PERMUTATIONS

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Abstract. We prove bijectively that the total number of cycles of all even permutations of $[n] = \{1, 2, \dots, n\}$ and the total number of cycles of all odd permutations of $[n]$ differ by $(-1)^n(n-2)!$, which was stated as an open problem by Miklós Bóna. We also prove bijectively the following more general identity:

$$\sum_{i=1}^{n} c(n,i) \cdot i \cdot (-k)^{i-1} = (-1)^k k!(n-k-1)!,$$

where $c(n,i)$ denotes the number of permutations of $[n]$ with i cycles.

1. Introduction

Let $c(n,i)$ denote the number of permutations of $[n] = \{1, 2, \dots, n\}$ with i cycles. The following equation is well known; for example see [1, 3]:

$$\sum_{i=1}^{n} c(n,i) x^i = x(x+1)\cdots(x+n-1). \tag{1}$$

Let n and k be positive integers with $k < n$. By differentiating (1) with respect to x and substituting $x = -k$, we get the following:

$$\sum_{i=1}^{n} c(n,i) \cdot i \cdot (-k)^{i-1} = (-1)^k k!(n-k-1)!. \tag{2}$$

In particular, if $k = 1$, then (2) implies the following theorem.

Theorem 1. *The total number of cycles of all even permutations of $[n]$ and the total number of cycles of all odd permutations of $[n]$ differ by $(-1)^n(n-2)!$.*

The problem of finding a bijective proof of Theorem 1 was proposed by Miklós Bóna and it has been added to [2] as an exercise (private communication with Richard Stanley and Miklós Bóna). In this note, we prove Theorem 1 bijectively by finding a sign-reversing involution. We also prove (2) bijectively.

2. Bijective proofs

Recall the lexicographic order on the pairs of integers, that is, $(i_1, j_1) \le (i_2, j_2)$ if and only if $i_1 < i_2$, or $i_1 = i_2$ and $j_1 \le j_2$. Note that this is a linear order.

Let $T(n)$ denote the set of pairs (π, C) where π is a permutation of $[n]$ and C is a cycle of π. Then Theorem 1 is equivalent to the following:

$$\sum_{(\pi,C)\in T(n)} \mathrm{sign}(\pi) = (-1)^n(n-2)!. \tag{3}$$

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Proof of Theorem 1. We define a map $\phi : T(n) \to T(n)$ as follows. Let $(\pi, C) \in T(n)$.

Case 1: C contains at most $n-2$ integers. Let (i,j) be the smallest pair in lexicographic order for distinct integers i and j which are not contained in C. Then we define $\phi(\pi, C) = (\tau_{ij}\pi, C)$, where τ_{ij} is the transposition exchanging i and j.

Case 2: C contains at least $n-1$ integers. If C does not contain 1, then we define $\phi(\pi, C) = (\pi, C)$. If C contains 1, then we have either $\pi = (a_0)(1, a_1, a_2, \ldots, a_{n-2})$ or $\pi = (1, a_0, a_1, \ldots, a_{n-2})$ in cycle notation for some integers a_i. Let $\pi' = (1, a_0, a_1, \ldots, a_{n-2})$ if $\pi = (a_0)(1, a_1, a_2, \ldots, a_{n-2})$, and $\pi' = (a_0)(1, a_1, a_2, \ldots, a_{n-2})$ if $\pi = (1, a_0, a_1, \ldots, a_{n-2})$. We define $\phi(\pi, C) = (\pi', C')$, where C' is the cycle of π' containing 1.

Let us define the *sign* of $(\pi, C) \in T(n)$ to be $\mathrm{sign}(\pi)$. It is easy to see that ϕ is a sign-reversing involution on $T(n)$ whose fixed points are precisely those $(\pi, C) \in T(n)$ such that 1 forms a 1-cycle and the rest of the integers form an $(n-1)$-cycle, which is C. Since there are $(n-2)!$ such fixed points of ϕ which all have sign $(-1)^n$, we get (3), and thus Theorem 1. ☐

Now we will generalize this argument to prove (2).

Let $P(n,k)$ denote the set of triples (π, C, f) where π is a permutation of $[n]$, C is a cycle of π and f is a function from the set of cycles of π except C to $[k]$. The left-hand side of (2) is equal to

$$\begin{aligned}\sum_{(\pi,C)\in T(n)} (-k)^{cyc(\pi)-1} &= \sum_{(\pi,C)\in T(n)} (-1)^{cyc(\pi)-1} k^{cyc(\pi)-1}\\ &= (-1)^{n-1} \sum_{(\pi,C,f)\in P(n,k)} \mathrm{sign}(\pi),\end{aligned}$$

because $\mathrm{sign}(\pi) = (-1)^{n-cyc(\pi)}$ and for given $(\pi, C) \in T(n)$, there are $k^{cyc(\pi)-1}$ choices of f with $(\pi, C, f) \in P(n,k)$. Thus we get that (2) is equivalent to the following:

$$\sum_{(\pi,C,f)\in P(n,k)} \mathrm{sign}(\pi) = (-1)^{n-k-1} k!(n-k-1)!. \tag{4}$$

Let us define the *sign* of $(\pi, C, f) \in P(n,k)$ to be $\mathrm{sign}(\pi)$. Let $\mathrm{Fix}(n,k)$ denote the set of elements $(\pi, C, f) \in P(n,k)$ such that (1) each integer $i \in [k]$ forms a 1-cycle of π and the integers $k+1, k+2, \ldots, n$ form an $(n-k)$-cycle of π, which is C and (2) the f values of the cycles of π except C are all distinct. Then, to prove (4), it is sufficient to find a sign-reversing involution on $P(n,k)$ whose fixed point set is $\mathrm{Fix}(n,k)$.

We will define a map $\psi : P(n,k) \to P(n,k)$ as follows. Let $(\pi, C, f) \in P(n,k)$.

Case 1: There is a pair (i,j) of integers $i < j$ such that $i \in C_1 \neq C$ and $j \in C_2 \neq C$ with $f(C_1) = f(C_2)$. Here we may have $C_1 = C_2$. Let (i,j) be the smallest such pair in lexicographic order. Then we define $\psi(\pi, C, f) = (\tau_{ij}\pi, C, f')$, where $f'(C') = f(C')$ if $i, j \notin C'$, and $f'(C') = f(C_1)$ otherwise. As before, τ_{ij} is the transposition exchanging i and j.

Case 2: Case 1 does not hold. Then the cycles of π except C are all 1-cycles whose f values are all distinct. Thus there are at most k 1-cycles of π except C.

We can represent (π, C, f) as a digraph D with vertex set $[n]$ as follows. For each integer i contained in C, add an edge $i \to \pi(i)$. For each integer i of $[n]$ which is not contained in C, add an edge $i \to f(i)$, where $f(i)$ is the f value of the 1-cycle

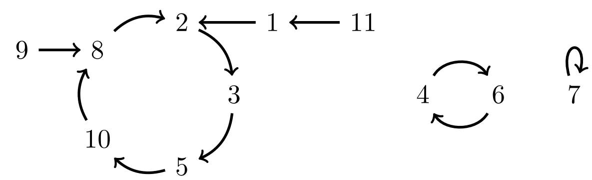

FIGURE 1. The digraph representing $(\pi, C, f) \in P(11, 8)$, where $\pi = (2,3,5,10,8)(1)(4)(6)(7)(9)(11)$, $C = (2,3,5,10,8)$, $f(1) = 2$, $f(4) = 6$, $f(6) = 4$, $f(7) = 7$, $f(9) = 8$ and $f(11) = 1$.

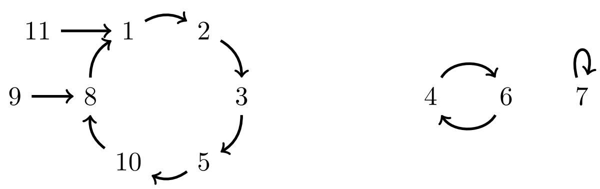

FIGURE 2. The digraph representing $(\pi, C, f) \in P(11, 8)$, where $\pi = (1,2,3,5,10,8)(4)(6)(7)(9)(11)$, $C = (1,2,3,5,10,8)$, $f(4) = 6$, $f(6) = 4$, $f(7) = 7$, $f(9) = 8$ and $f(11) = 1$.

(i) consisting of i. For example, see Figures 1 and 2. Note that we can recover (π, C, f) from D even when D consists of cycles only because in this case C is the only cycle containing integers greater than k.

Now we consider the two sub-cases where C contains an integer in $[k]$ or not.

Sub-Case 2-a: C does not contain any integer in $[k]$. It is easy to see that we have this sub-case if and only if $(\pi, C, f) \in \mathrm{Fix}(n, k)$. We define $\psi(\pi, C, f) = (\pi, C, f)$.

Sub-Case 2-b: C contains an integer in $[k]$. Let m be the smallest such integer.

For an integer $i \in C$, we say that i is *free* if $i \in [k]$ and the in-degree of i in D is 1, i.e. there is no integer outside of C pointing to i. A sequence $(m_1, m_2, \dots, m_\ell)$ of integers in C is called a *free chain* if it satisfies (1) for each $i \in [\ell] \setminus \{1\}$, m_i is free and $m_i = \pi(m_{i-1})$, and (2) for each $i \in [\ell]$, m_i is the ith-smallest integer in C. Note that we always have a free chain, for example the sequence consisting of m alone. Moreover, there is a unique maximal free chain.

Let $(m_1, m_2, \dots, m_\ell)$ be the maximal free chain. Let $\overline{m} = m_1$ if ℓ is odd, and $\overline{m} = m_2$ if ℓ is even.

Example 1. The maximal free chains of the digraphs in Figures 1 and 2 are $(2,3,5)$ and $(1,2,3,5)$ respectively. Thus $\overline{m} = 2$ in both Figures 1 and 2.

Let D' be the digraph obtained from D by doing the following. If $\overline{m}$ is free, then let u, v be the integers in C such that D has the edges $v \to u$ and $u \to \overline{m}$. It is not difficult to see that in this case C has at least two integers, which implies $u \neq \overline{m}$. Then we remove the edge $v \to u$ and add an edge $v \to \overline{m}$. If $\overline{m}$ is not free, then let u and v be the integers with $u \notin C$ and $v \in C$ such that D has the edges $u \to \overline{m}$ and $v \to \overline{m}$. Then we remove the edge $v \to \overline{m}$ and add an edge $v \to u$.

We define $\psi(\pi, C, f)$ to be the element in $P(n, k)$ represented by D'.

Example 2. Let (π, C, f) be represented by the digraph in Figure 1. Since $\overline{m} = 2$, $\psi(\pi, C, f)$ is represented by the digraph in Figure 2. Note that $\psi(\psi(\pi, C, f)) = (\pi, C, f)$.

It is easy to see that ψ is a sign-reversing involution on $P(n, k)$ with fixed point set $\mathrm{Fix}(n, k)$. Thus we have proved (2) bijectively.

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